Exposé

- G ($= \mathbb{G}(\mathbb{R})$, $\mathbb{G}/_{\mathbb{Q}}$ reductif (semi simple))
- $K \subset G$ maximal compact
- $\Gamma \subseteq G$ discret (ex : $\Gamma \subset \mathbb{G}(\mathbb{Q})$ sous groupe de congruence)

~~Thm~~ . ~~Hyp : Vol(${}_{\Gamma}\backslash G$) finie~~

- $X = G/K$

1) ~~Vol(${}_{\Gamma}\backslash G$) ...~~

$\boxed{\text{Thm}}$ $H^{*}_{(2)}(\mathrm{II}^{\Gamma}_{G}) \xrightarrow{\varphi} H^{*}_{(2)}({}_{\Gamma}\backslash X, \mathbb{R}) \longrightarrow H^{*}({}_{\Gamma}\backslash X, \mathbb{R}) = H^{*}(\Gamma, \mathbb{R})$

cohom formes de formes de Rham localement constantes, L^2

cohom de de Rham L^2

cohomology de de Rham

Isom si $\underline{{}_{\Gamma}\backslash X \text{ compact}}$

φ est un isom si $* \le m(\mathbb{G})$

$\boxed{\text{rem}}$ si $\mathrm{Vol}({}_{\Gamma}\backslash G) = \text{fini}$, $\mathbb{H}_{(\mathbb{R})}$ $H^{*}_{(2)}(\mathrm{I}^{\Gamma}_{G}) = 0$

I) (g, K)-modules

def : $V \in (g, K)$-mod si V est un $\mathbb{R}$-ev.

- ~~g-module~~ $g \curvearrowright V$, $K \curvearrowright V$
- $V = \bigcup$ rep de dim finies de K.
- $k \in \mathfrak{k}$, $k.v = \lim_{t \to 0} \frac{\exp(tk)v - v}{t}$
- $k \in K$, $(Ad(k) \cdot g)\, v =$ ~~Ad~~ $k \cdot g \cdot k^{-1} v$.

$V \in (g, K)$-mod, $H^{*}(g, K, V) = \mathrm{Ext}^{*}_{(g,K)}(\mathbb{R}, V)$

dérivé de $H^{0}(g, K, V) = (V)^{g, K}$

$\boxed{\text{Calcul explicite}}$ via.

$C^{*}(g, K, V)$

$C^{q}(g, K, V) = \mathrm{Hom}_{K}(\Lambda^{q} g/\mathfrak{k}, V)$

$d : C^{q} \longrightarrow C^{q+1}$

$f \longrightarrow df(X_0, \dots, X_q) = \Sigma(-1)^{i} X_i \cdot f(X_0, \dots \widehat{X_i} \dots X_q)$

$$\mathrm{Rep}(G) \longrightarrow (\mathfrak{g},K)\text{-mod}$$

$$V \longrightarrow HC(V) = \{v \in V,\ K\text{-f}\ C^\infty,\ K\text{-finis}\}$$

$$\boxed{H^*(\mathfrak{g},K,V) := H^*(\mathfrak{g},K,HC(V))}$$

<u>Ex</u> dans $\mathrm{Rep}(G)$:

1) $V_1 = LC(\Gamma\backslash G, \mathbb{R})$

2) $V_2 = L^2(\Gamma\backslash G, \mathbb{R})$

3) $V_3 = C^\infty(\Gamma\backslash G, \mathbb{R})$

- $HC(V_2) \to HC(V_3)$

$\boxed{\text{si } \mathrm{vol}(\Gamma\backslash G) < \infty}$

- $V_1 \to V_2$

Prop

$$\begin{array}{ccccc} H^*(\mathfrak{g},K,V_1) & \to & H^*(\mathfrak{g},K,V_2) & \to & H^*(\mathfrak{g},K,V_3) \\ \| & & \| & & \wr\| \\ H^*(I^\Gamma_G) & \longrightarrow & H^*_{(2)}(\Gamma\backslash X, \mathbb{R}) & \longrightarrow & H^*(\Gamma\backslash X, \mathbb{R}) \end{array}$$

démo :

$$C^*(\mathfrak{g},K,V_3) \overset{\text{canonique}}{\equiv} \Omega^*_{dR}(\Gamma\backslash X, \mathbb{R}) \quad /\ \text{de Rham } C^\infty.$$

- $C^0(\mathfrak{g},K,V_3) = \mathrm{Hom}_K(1, \mathcal{C}^\infty(\Gamma\backslash G, \mathbb{R}))$
  $= \mathcal{C}^\infty(\Gamma\backslash X, \mathbb{R})$

- $C^1(\mathfrak{g},K,V_3) = \mathrm{Hom}_K(\Lambda\, \mathfrak{g}/\mathfrak{k}, \mathcal{C}^\infty(\Gamma\backslash G, \mathbb{R})) \overset{?}{=} \Omega^1(\Gamma\backslash X, \mathbb{R})$

- $C^\infty(G,\mathbb{R}) \otimes \mathfrak{g}^* \xrightarrow{\sim} \Omega^1(G,\mathbb{R})$

  $L_G \otimes 1 \longleftrightarrow L_G$

  $\left[X \in \mathfrak{g} \mapsto f \mapsto \left(g \mapsto \lim_{t\to 0} \dfrac{f(g\exp tX) - f(g)}{t}\right)\right]$

  $R_G \otimes \mathrm{Ad}^*_G \longleftrightarrow R_G$

- $C^\infty(\Gamma\backslash G, \mathbb{R}) \otimes \mathfrak{g}^* \xrightarrow{\sim} \Omega^1(\Gamma\backslash G, \mathbb{R})$

  $\uparrow \qquad\qquad \uparrow$

  $C^\infty(\Gamma\backslash G, \mathbb{R}) \otimes (\mathfrak{g}/\mathfrak{k})^* \xrightarrow{\sim} \pi^*\Omega^1(\Gamma\backslash X, \mathbb{R})$

$\Gamma\backslash G \xrightarrow{\pi} \Gamma\backslash X$

- $\mathrm{Hom}_K(\mathfrak{g}/\mathfrak{k}, \mathcal{C}^\infty(\Gamma\backslash G, \mathbb{R})) \xrightarrow{\sim} \Omega^1(\Gamma\backslash X, \mathbb{R})$

Theorie de Hodge des $(\mathfrak{g},K)$-module

$\boxed{\text{Hyp}}$ V est muni d'un produit scalaire invariant
[$V \simeq V^\vee$ au sens de $(\mathfrak{g},K)$-modules]

$\boxed{\text{Ex}}$ $L^2_\Gamma(G, \mathbb{R})$

$\boxed{\text{rem}}$ $T(X, \mathbb{R}) = \mathrm{Hom}_K(\mathbb{1}, C^\infty_\Gamma(G) \otimes \cancel{\Lambda}\, \mathfrak{g}/\mathfrak{k})$

$\cancel{\mathrm{Hom}_K(\mathbb{1}, \Lambda\mathfrak{g}/\mathfrak{k})}$

$B|_{\mathfrak{g}/\mathfrak{k}}$

$(\mathrm{Sym}^2\, T(X,\mathbb{R}))^* \cong \mathrm{Hom}_K(\mathbb{1}, (\mathrm{Sym}^2 \mathfrak{g}/\mathfrak{k})^*) \ni B$

$V \otimes (\Lambda^q \mathfrak{g}/\mathfrak{k})^\vee$

produit scalaire K-invariant

$\boxed{\text{Prop}}$ $d: C^q(\mathfrak{g},K,V) \to C^{q+1}(\mathfrak{g},K,V)$

possède un adjoint $\partial: C^{q+1}(\mathfrak{g},K,V) \to C^q(\mathfrak{g},K,V)$

- $\Delta = d\partial + \partial d$, $\ker \Delta =$ ~~formes~~ $\ker d \cap \ker \partial$.

$$C^q(\mathfrak{g},K,V) \hookrightarrow \Lambda^q \mathfrak{p}^* \otimes V \simeq \Lambda^{d_q} \mathfrak{p}^* \otimes V$$

$$\mu = \sum \mu_I\, \omega^I \qquad \eta = \sum \eta_I \cdot \omega^I$$

$$\langle \mu, \eta \rangle = \sum (\mu_I, \eta_I)$$

$*: V \otimes \Lambda^q \mathfrak{g}/\mathfrak{k} \to$

$\alpha \in V \otimes \Lambda^q$, $\beta \in V \otimes \Lambda^q$ $\quad V \otimes \Lambda^{d-q} \mathfrak{g}/\mathfrak{k}$

$\langle \alpha, * \beta \rangle =$

$\langle \alpha, \beta \rangle = \langle \alpha, *\beta \rangle$

- $C \in Z(\mathfrak{g})$: $\mathbb{1} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}^* \xrightarrow{1 \otimes B} \mathfrak{g} \otimes \mathfrak{g} \longrightarrow U(\mathfrak{g})$

$V \otimes \Lambda^q \times V \otimes \Lambda^q$

$C = \mathrm{Im}(\cancel{\mathbb{1}}\ 1 \in \mathbb{1}) \in Z(\mathfrak{g})$

$\Lambda \otimes \Lambda^q \times$

$V \otimes \Lambda^{d-q}$

- $\Delta \curvearrowright C^*(\mathfrak{g},K,V)$
$\Delta = -C$

$\boxed{\text{Corollaire}}$ Supposons que $C = \lambda\, \mathrm{Id}$ sur V.

Si $\lambda \neq 0$ $H^*(\mathfrak{g},K,V) = 0$

$\boxed{\text{preuve}}$. $\Delta = -\lambda\, \mathrm{Id}$ sur $H^*(\mathfrak{g},K,V)$.

. $\Delta = 0$ sur $H^*(\mathfrak{g},K,V)$

. Mais $\Delta \underset{\text{homotope}}{\sim} 0 \Rightarrow \Delta = 0$ sur $H^*(\mathfrak{g},K,V)$
sur $C^*(\mathfrak{g},K,V)$

Applicat° 1 — Calcul de la cohomologie du dual compact

Hyp: G, K connexes

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$$
$$\check{\mathfrak{g}} = i\mathfrak{p} + \mathfrak{k} \longleftrightarrow \check{G} \hookleftarrow K$$

forme compact de G

$$\check{X} = \check{G}/K$$

Prop: $H^*(\check{X}, \mathbb{R}) = \left(\Lambda^q \mathfrak{p}^*\right)^K \quad \left(= H^*(I_G^{\Gamma})\right)$

preuve: $H^*(\check{X}, \mathbb{R}) = H^*(\check{\mathfrak{g}}, K, \mathcal{C}^\infty(\check{G}, \mathbb{R}))$

· Peter Weyl: $L^2(\check{G}, \mathbb{R}) = \widehat{\bigoplus_{\lambda = \text{plus haut poids}}} m(\lambda) V_\lambda$ — repr de dim finie de $\check{G}$

· Theorie de Hodge: $H^*(\check{\mathfrak{g}}, K, \mathcal{C}^\infty(\check{G}, \mathbb{R})) = H^*(\check{\mathfrak{g}}, K, \mathcal{C}(\check{G}, \mathbb{R})[C=0])$

$$= H^*(\check{\mathfrak{g}}, K, \mathbb{R})$$

$C = (\|\lambda+\rho\| - \|\rho\|) \cdot Id$ sur V_λ

$$= \left(\Lambda^q (\mathfrak{g}^*/\mathfrak{p})\right)^K$$

Applicat° 2 (thm de Garland)

Th: V $(\mathfrak{g}, K)$-module irred, $C = \lambda\, Id$ sur V, et $H^0(\mathfrak{g}, K, V) = 0$. Alors $H^q(\mathfrak{g}, k, V) = 0 \quad \forall q \leq m(\mathfrak{g})$

Corollaire: $V = HC(L^2_0(\Gamma \backslash G, \mathbb{R}))[C=0]$ si $\mathrm{vol}(G/\Gamma) = +\infty$

$V = HC(L^2(\Gamma \backslash G, \mathbb{R}))[C=0] / HC(LC(\Gamma \backslash G, \mathbb{R}))$ si $\mathrm{vol}(G/\Gamma) < +\infty$

$\forall q \geq 0 \cdot H^0(\mathfrak{g}, K, V) = 0$.

$H^*(I_G^\Gamma) \longrightarrow H^*_{(2)}(\Gamma \backslash X, \mathbb{R})$ iso si $* \leq m(\mathfrak{g})$

preuve

$$H^*(\mathfrak{g}, K, LC^2(\Gamma \backslash G, \mathbb{R})) \longrightarrow H^*(\mathfrak{g}, K, L^2(\Gamma \backslash X, \mathbb{R})) \longrightarrow$$

$$H^*(\mathfrak{g}, K, LC^2(\Gamma \backslash G, \mathbb{R})[C=0]) \to H^*(\mathfrak{g}, K, L^2(\Gamma \backslash X, \mathbb{R})[C=0])$$

Thm : V $(\mathfrak{g}, K)$ module unitaire. $C = \lambda\, \mathrm{Id}$ sur V

et $H^0(\mathfrak{g}, K, V) = 0$. Alors $H^q(\mathfrak{g}, K, V) = 0$ si $q \leq m(\mathfrak{g})$

~~Preuve~~ Corollaire . Si $\mathrm{Vol}(\Gamma\backslash G) = +\infty$, $V = L^2(\Gamma\backslash G, \mathbb{R})[C=0]$

$$H^*_{(2)}(\Gamma\backslash X, \mathbb{R}) = H^*(\mathfrak{g}, K, V)$$
$$= H^*(\mathfrak{g}, K, V[C=0])$$

Corollaire Si $\mathrm{Vol}(\Gamma\backslash G) < +\infty$

$$V = L^2(\Gamma\backslash G, \mathbb{R}) \big/ LC(\Gamma\backslash G, \mathbb{R}) \quad [C=0]$$

$$H^q(\mathfrak{g}, K, V) = 0 \text{ si } q \leq m(\mathfrak{g})$$

$$\Rightarrow \quad H^q(\mathfrak{g}, K, LC(\Gamma\backslash G, \mathbb{R})) \to H^q(\mathfrak{g}, K, L^2(\Gamma\backslash G, \mathbb{R}))$$
iso si $q \leq m(\mathfrak{g})$

rem En general, il y a plusieurs $(\mathfrak{g}, K)$ modules irred dans $L^2(\Gamma\backslash G, \mathbb{R})$ avec $C = 0$!! contrairement à $L^2(\check{G}, \mathbb{R})$

$$\neq LC(\Gamma\backslash G, \mathbb{R})$$

Ex :

$G = SL_2(\mathbb{R})$. Γ congruence.

$S_2(\Gamma)$: espaces des formes modulaires cuspidales de niveau Γ, de poids 2.

$S_{2,s}(\Gamma)$

$$\{ f : ~~\mathcal{H}~~ = \mathcal{H} \to \mathbb{C} \;,\; f(\gamma z) = (cz+d)^2 f(z) \}$$
$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

f holomorphe
cuspidale.

$$\Phi_f(g) = f(g(i))\,(ci+d)^{-2} \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$g(i) \in \mathcal{H}$

$\Phi_f \in L^2(\Gamma\backslash G, \mathbb{R})$

$$\Phi_f(g\,k(\theta)) = e^{i2\theta}\,\Phi_f(g)$$

rem

hol.

$\mathfrak{p} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$

$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$

$= ~~\mathfrak{p}_{\mathbb{C}}~~ \cong \mathbb{C}H_+ \oplus \mathbb{C}H_-$

$\sigma\big(\begin{smallmatrix} 1 & \\ & -1 \end{smallmatrix}$

$f(z)\,dz \in H^1_{(2)}(\Gamma\backslash X, \mathbb{R})$ $=$ $\boxed{\Phi_f \quad H^*_+}$ $\in C^1(\mathfrak{g}, K, L^2(\Gamma\backslash G, \mathbb{R}))$

$H^1_{(2)}(\Gamma\backslash X, \mathbb{R})$

$\boxed{SL_2}$ $H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in k$

$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in p$

$[H, X_1] = -2X_2$, $[H, X_2] = 2X_1$, $[X_1, X_2] = 2H$

$C = X_1^2 + X_2^2 - H^2$

$\cdot\, \eta = u_1 X_1^* + u_2 X_2^* \in C^1(g, K, V)$ $\Leftrightarrow$ $H.u_1 - 2u_2 = 0$, $Hu_2 + 2u_1 = 0$

on veut mq $\boxed{g \cdot u_i = 0}$

$\cdot$ $C = 0$ sur V : $\|X_1 u_i\|^2 + \|X_2 u_i\|^2 = \|H u_i\|^2$

$\cancel{\|X_2 u_2\|^2 = H}$

$\Rightarrow$ $k . u_i = 0 \Leftrightarrow p . u_i = 0$

$\varphi(u) = \|[X_1, X_2] u_1\|^2 + \|[X_1, X_2] u_2\|^2$

$\varphi(u) = 4\|H u_1\|^2 + 4\|H u_2\|^2$.

on veut mq $\varphi(u) = 0$.

$\varphi(u) = 8 \langle X_2 u_2, X_1 u_1 \rangle - 8 \langle X_1 u_2, X_2 u_2 \rangle$

preuve $\|[X_1, X_2] u_2\|^2 = \langle [X_1, X_2] u_2, [X_1, X_2] u_2 \rangle$

$= \langle 2H u_2, X_1 X_2 u_2 \rangle - \langle 2H u_2, X_2 X_1 u_2 \rangle$

$= \langle 4 u_2, X_1 X_2 u_2 \rangle - \langle 4 u_2, X_2 X_1 u_2 \rangle$

$= -\langle 4 X_1 u_2, X_2 u_2 \rangle + 4 \langle X_2 u_2, X_1 u_2 \rangle$

$\rightsquigarrow$ $\overset{\lambda}{\overset{\|}{4 \sum_{i,j} \|X_i u_j\|^2}} - 8 \langle X_2 u_2, X_1 u_1 \rangle + 8 \langle X_1 u_2, X_2 u_2 \rangle = 0$

On considère la forme quadratique F sur $p \otimes p \otimes V = \oplus$

$x = \sum x_{i,j} X_i \otimes X_j$

$\Leftarrow$ $\sum (x_{i,j})^2 - 8 x_{2,2} \cdot x_{11} + 8 x_{12} \cdot x_{22}$

$F_V = F \otimes \langle\ \rangle_V$ sur $p \otimes p \otimes V = \oplus V X_i \otimes X_j$

$\lambda(u) = F_V((x_i, u_j))$ $\|$ λ $F_V \geq 0$ $\Leftrightarrow$ $F > 0$.

~~en~~ général.